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学术论文

The Complexity of Symmetric #CSP on Minor-closed Graph Classes

对称#CSP 在图子式封闭图上的复杂性

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Minor: A graph G' is a minor of G if G' can be obtained from G by repeatedly deleting vertices, deleting edges and merging two incident vertices into one.

Minor-closed graph class: For any finite set H of graphs, we define the graph class $fb(H) = \{G|$ no graph in H is a minor of $G\}$,and H is said to be the forbidden minor set of it.

One of the most effective ways to characterize graph classes

#CSP problem on minor-closed graphs: Parameterized by a function set F and a forbidden minor set H, denoted as #CSP(F)[H]

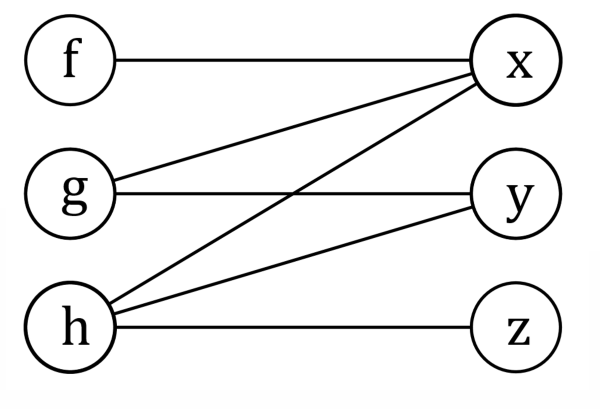
#CSP(*F*)[*H*]:
Input: A bipartite graph $G = (U, V, E) \in fb(H)$ with $|U| = m, |V| = n$. Each $u \in U$ is assigned a signature f_u from F and each $v \in V$ represent a variable x_v, and $uv \in E$ if and only if f_u depends on x_v.

Output:

$$Z(I) = \sum_{x_1,\ldots,x_n \in \{0,1\}} \prod_{1 \le i \le m} f_i(x_{i_1}, \ldots, x_{i_k})$$

Sym-#CSP(*F*)[*H*]:#CSP(F)[H] with each signature in F being symmetric, which means the value of each $f \in F$ only depends on the number of 1’s in its input.

An example of the input of #CSP. f, g, h represent signatures while x, y, z represent variables.



#CSP problem is one of the fundamental framework in counting problems. Many counting problems, such as #SAT, counting graph homomorphisms, can be expressed in the form of #CSP.

Our main results on Sym-#CSP(*F*)[*H*]

Forbidden minor	K_4	K_5	K_6	K_7	K_8
If $F \subseteq \mathscr{A}$ or $F \subseteq \mathscr{P}$	P-time [Cai, Lu, Xia 2014]				
Else if $F \subseteq \mathscr{M}$-trans	P-time(1)	P-time(2)	#P-hard(3); P-time(4)		#P-hard(5)
Else		#P-hard [Guo, Williams 2020]			

A summary of our results, which is labelled in red and numbered by (1)-(5). Each row denotes a certain case of F, where ``$\mathscr{A}$’, ``$\mathscr{P}$’ and ``$\mathscr{M}$-trans’’ are the only three tractable cases in #CSP on planar graphs. Each column denotes the graph that $H$ contains, where $K_n$ denote the complete graph with $n$ vertices. The symbol ``P-time'' denotes the corresponding problem is polynomial time computable, while ``#P-hard '' denotes the corresponding problem is #P-hard. In particular, ``#P-hard, P-time'' denotes the corresponding problem is #P-hard in general, but has a polynomial time algorithm when the degree of each vertex is upper bounded.

Significance

1.Present a complete characterization of the complexity of sym-#CSP(F)[K_n].
2. Demonstrate that whether the degree of each vertex is upper bounded influence the complexity.
3. Develop a systematic approach for analyzing the complexity of sym-#CSP on minor-closed graph classes.

Proof sketch

Proof sketch of (1)

Condition: G forbid K_4 minor → 1.Obtain a tree decomposition with width at most 2 → 2.Obtain balanced vertex separators → 3.Perform a divide and conquer algorithm in polynomial time

Proof sketch of (2)-(5)

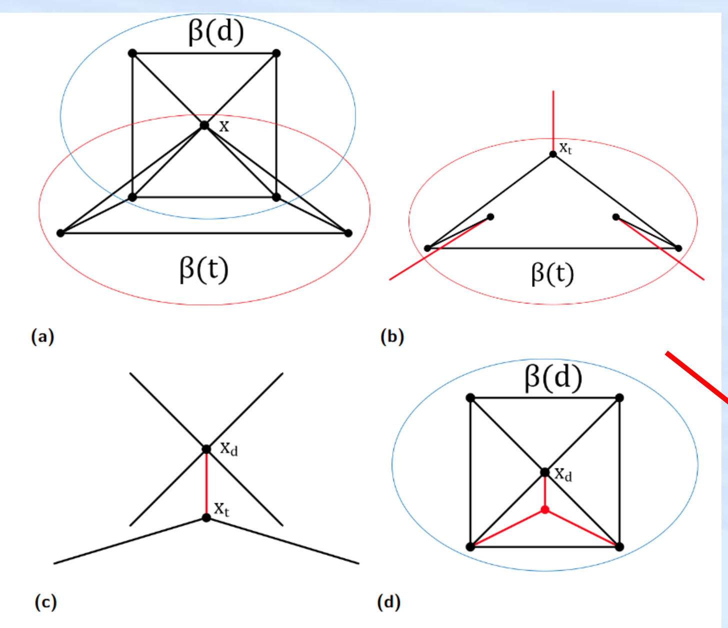
Condition:$F \subseteq \mathscr{M}$-trans → Holographic transformation → Obtain an equivalent form of Holant($\hat{F}|\widehat{EQ}$) of the problem. Here, $\hat{F} \cup \widehat{EQ} \subseteq \mathscr{M}$, and there exists $f \in F$ satisfying $f \in \mathscr{M}$-$\mathscr{A}$.

Algorithm part

Cases: (2)(4)

(2) Defined in [Curticapean 2014]

(4) Defined in [Thilikos, Wiederrecht 2014]



1.Obtain a specific tree decomposition;
2.Compute the value of a leaf and record it;
3.Replace the leaf with a single vertex;
4.Doing Step 2,3 successively.

Hardness part

Cases: (3)(5)

(3) Counting matchings on planar graphs [Xia, Zhang, Zhao 2007]

(5) Counting perfect matchings on 3-regular graphs [Dagum, Luby 1992]

1.Start from a specific #P-hard problem;
2. Reduce the problem to an intermediate problem whose underlying graph forbids the target minor;
3. Simulate signatures in the intermediate problem in a planar way.